05/03/18

1 **A**

$$\lim_{n\to+\infty} \frac{n^2}{2^{\alpha n}+5}\cdot\frac{3^n+7}{\sqrt[4]{n+2}-n^{1/4}} \qquad \alpha\in\mathbb{R}^+$$

$$= \lim_{n\to+\infty} \frac{n^2}{n^{1/4}\left(\sqrt[4]{1+\frac{2}{n}}-1\right)}\cdot\left(\frac{3}{2^\alpha}\right)^n$$

$$= \lim_{n\to+\infty} \frac{n^2}{n^{1/4}\cdot\left(\cancel{1}+\frac{1}{2n}+o\left(\frac{1}{n}\right)\cancel{-1}\right)}\cdot\left(\frac{3}{2^\alpha}\right)^n$$

$$= \lim_{n\to+\infty} 2\,n^{2+1-1/4}\cdot\left(\frac{3}{2^\alpha}\right)^n =$$

$$= \lim_{n\to+\infty} 2\,n^{11/4}\left(\frac{3}{2^\alpha}\right)^n = \begin{cases} 0 & \text{se } \frac{3}{2^\alpha}<1 \iff 2^\alpha>3 \iff \boxed{\alpha>\log_2 3} \\ +\infty & \text{se } \frac{3}{2^\alpha}\geq 1 \iff \boxed{\alpha\leq\log_2 3} \end{cases}$$

B

$$\lim_{n\to+\infty} \frac{2^n+1}{n}\cdot\frac{\sqrt[3]{n+1}-n^{1/3}}{2+3^{\beta n}} =$$

$$= \lim_{n\to+\infty} \left(\frac{2}{3^\beta}\right)^n\cdot\frac{n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3}-1\right]}{n} =$$

$$= \lim_{n\to+\infty} \left(\frac{2}{3^\beta}\right)^n\cdot\frac{n^{1/3}\cdot\frac{1}{3n}}{n}$$

$$= \lim_{n\to+\infty} \left(\frac{2}{3^\beta}\right)^n\cdot\frac{1}{3n^{5/3}} = \begin{cases} 0 & \text{se } \frac{2}{3^\beta}\leq 1 \iff 3^\beta\geq 2 \iff \boxed{\beta\geq\log_3 2} \\ +\infty & \text{se } \boxed{\beta<\log_3 2} \end{cases}$$

(2) (A) $\lim\limits_{x \to 0} \dfrac{(e - e)[\sin(x+x^2) - e + \cos x + x^2]}{\sqrt{1+\dfrac{4}{x^8}} - \dfrac{2}{x^4}}$

$\boxed{\text{Num}}$: • $e^{x+1} - e = e[e^x - 1] = e(\cancel{1} + x + o(x) - \cancel{1})$

$= e \cdot x + o(x)$

• $\sin(x+x^2) = \cancel{x} + \cancel{x^2} - \dfrac{1}{6}x^3 + o(x^3)$

• $-e^{x+x^2} = -1 - x - x^2 - \dfrac{1}{2}(x+x^2)^2 - \dfrac{1}{6}x^3 + o(x^3)$

$= \cancel{-1} \cancel{-x} \cancel{-x^2} - \dfrac{1}{2}x^2 - x^3 - \dfrac{1}{6}x^3 + o(x^3)$

• $\cos x + x^2 = \cancel{1} - \dfrac{1}{2}x^2 + o(x^3) + x^2$

↳ Li sommo e rimane:

$\boxed{\text{Num}} = (ex + o(x)) \cdot \left[-\dfrac{1}{6}x^3 - \cancel{\dfrac{1}{2}x^2} - x^3 - \dfrac{1}{6}x^3 \cancel{-\dfrac{1}{2}x^2} + \cancel{x^2} + o(x^3) \right]$

$= (ex + o(x)) \cdot \left[\dfrac{-1-3}{3} x^3 + o(x^3) \right] =$

$= \boxed{-\dfrac{4}{3} e x^4 + o(x^4)}$

Denom: (N.B $\frac{1}{x^8} \longrightarrow +\infty$ per $x \to 0$!!.)

$$\sqrt{1+\frac{4}{x^8}} - \frac{2}{x^4} = \frac{2}{x^4}\left[\sqrt{\frac{x^8}{4}+1} - 1\right]$$

$$= \frac{\cancel{2}}{x^4}\left[\cancel{1} + \frac{1}{\cancel{2}}\cdot\frac{x^8}{4} \cancel{-1} + o(x^8)\right]$$

$$= \frac{1}{4}x^4 + o(x^4)$$

Concludendo:

$$\lim_{x\to 0} \frac{Num}{Den} = \lim_{x\to 0} \frac{-\frac{4}{3}e\,x^4}{\frac{1}{4}x^4} = \boxed{-\frac{16}{3}e}$$

(B) $$\lim_{x\to 0} \frac{\left[\tan\left(x-\frac{1}{2}x^2\right) - \log\left(1+x-x^2\right) - \sin(x^2)\right]\left(e^2 - e^{2+x}\right)}{\frac{1}{x^4} - \sqrt{1+\frac{1}{x^8}}}$$

Num: $e^2 - e^{2+x} = e^2\left[1 - e^x\right] = e^2\left(1 - 1 - x + o(x)\right)$

$$= -e^2 x + o(x)$$

$$\sin\left(x - \tfrac{1}{2}x^2\right) = \cancel{x} - \cancel{\tfrac{1}{2}x^2} + \tfrac{1}{3}x^3 + o(x^3)$$

$$-\log(1+x-x^2) = -x + x^2 + \frac{1}{2}(x-x^2)^2 - \frac{1}{3}x^3 + o(x^3)$$

$$= \cancel{-x} \cancel{+x^2} + \cancel{\frac{1}{2}x^2} - x^3 - \frac{1}{3}x^3 + o(x^3)$$

$$-\sin(x^2) = \cancel{-x^2} + o(x^3)$$

Li sommo e ottengo:

$$\underline{\text{Num}} = \left(\cancel{\frac{1}{3}x^3} - x^3 - \cancel{\frac{1}{3}x^3} + o(x^3)\right)\left(-e^2 x + o(x)\right)$$

$$= + e^2 x^4 + o(x^4)$$

$$\underline{\text{Den}} : \quad \frac{1}{x^4} - \sqrt{1 + \frac{1}{x^8}} =$$

$$= \frac{1}{x^4}\left[1 - \sqrt{x^8+1}\right] = \frac{1}{x^4}\cdot\left(\cancel{1} - \cancel{1} - \frac{1}{2}x^8 + o(x^8)\right)$$

$$= -\frac{1}{2}x^4 + o(x^4)$$

Concludendo:

$$\lim_{x\to 0} \frac{\text{Num}}{\text{Den}} = \lim_{x\to 0} \frac{e^2 x^4}{-\frac{1}{2}x^4} = \boxed{-2e^2}$$

13) A) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{(\sin x+2)(1-\cos^2 x)}\,dx$

$$= \int_{\pi/4}^{\pi/2} \frac{\cos x}{(\operatorname{sen} x+2)\cdot \operatorname{sen}^2 x}\,dx$$

Pongo $t = \operatorname{sen} x$

$\Rightarrow dt = \cos x\, dx$

$$= \int_{\sqrt{2}/2}^{1} \frac{1}{(t+2)\cdot t^2}\,dt$$

Cerco A, B, C tali che:

$$\frac{1}{(t+2)\cdot t^2} = \frac{A}{t+2} + \frac{B}{t} + \frac{C}{t^2}$$

$$\Leftrightarrow \quad At^2 + Bt(t+2) + C(t+2) = 1$$

$$\Leftrightarrow \quad At^2 + Bt^2 + 2Bt + Ct + 2C = 1$$

$$\Leftrightarrow \quad t^2(A+B) + t(2B+C) + 2C = 1$$

$$\Leftrightarrow \begin{cases} A+B=0 \\ 2B+C=0 \\ 2C=1 \end{cases} \Leftrightarrow \begin{cases} A=-B \\ 2B+\frac{1}{2}=0 \\ C=\frac{1}{2} \end{cases} \quad \begin{cases} A=\frac{1}{4} \\ B=-\frac{1}{4} \\ C=\frac{1}{2} \end{cases}$$

$$\Rightarrow \int_{\sqrt{2}/2}^{1} \left(\frac{1}{4(t+2)} - \frac{1}{4t} + \frac{1}{2t^2}\right)dt =$$

$$\left[\frac{1}{4}\log(t+2) - \frac{1}{4}\log t - \frac{1}{2}\frac{1}{t}\right]_{\sqrt{2}/2}^{1} = \ldots$$

[L'altro è analogo!]

(4) $f(x) = \left| \frac{\log(x+2)}{x+2} \right|$

DOMINIO $\quad x+2>0 \Rightarrow D = (-2, +\infty)$

LIMITI

- $\lim_{x \to -2^+} \left| \frac{\overbrace{\log(x+2)}^{\to -\infty}}{\underbrace{x+2}_{\to 0}} \right| = +\infty$

- $\lim_{x \to +\infty} \left| \frac{\log(x+2)}{x+2} \right| = \emptyset$

SEGNO:

$f(x) \geq 0 \quad \forall x \in D$

$\underline{f(x) = 0} \iff \log(x+2) = 0$

$\iff x+2 = 1 \iff \boxed{x = -1}$

MONOTONIA:

$$f'(x) = \operatorname{sgn}\left(\frac{\log(x+2)}{x+2}\right) \cdot \frac{\frac{x+2}{x+2} - \log(x+2)}{(x+2)^2}$$

$$= \boxed{\operatorname{sgn}\left(\frac{\log(x+2)}{x+2}\right) \cdot \frac{1 - \log(x+2)}{(x+2)^2}}$$

Studio segno di f':

$$f'(x) \geq 0 \iff \operatorname{sgn}\left(\frac{\log(x+2)}{x+2}\right) \cdot \left(1 - \log(x+2)\right) \geq 0$$

- $1 - \log(x+2) \geq 0 \iff \log(x+2) \leq 1 \iff \boxed{x \leq e-2}$

- $\operatorname{sgn}\left(\frac{\log(x+2)}{x+2}\right) \geq 0 \iff \frac{\log(x+2)}{x+2} > 0$

$$\begin{cases} \log(x+2) > 0 \iff x+2 > 1 \iff x > -1 & \mathrm{I} \\ x+2 > 0 \iff x > -2 & \mathrm{II} \end{cases}$$

fuori da D

	-2		-1	
I		$-$		$+$
II		$+$		$+$
		$-$		$+$

$$\longrightarrow \operatorname{sgn}\left(\frac{\log(x+2)}{x+2}\right) = \begin{cases} -1 & \text{se } -2 < x < -1 \\ +1 & \text{se } x > -1 \end{cases}$$

Concludendo:

$\boxed{f'(x) \geq 0} \iff$

$e-2 \sim 0{,}7$

-2		-1	0	$e-2$	
	$+$		$+$		$-$
	$-$		$+$		$+$
	$-$		$+$		$-$

Quindi $f \searrow$ in $(-2, -1)$

$f \nearrow$ in $(-1, e-2)$

$f \searrow$ in $(e-2, +\infty)$

PUNTI di NON DERIVABILITA':

Il punto in cui eventualmente ci possono essere problemi di non derivabilità è quello in cui si annulla l'argomento del valore assoluto, ovvero il punto x t.c.

$$\log(x+2) = 0 \iff x = -1$$

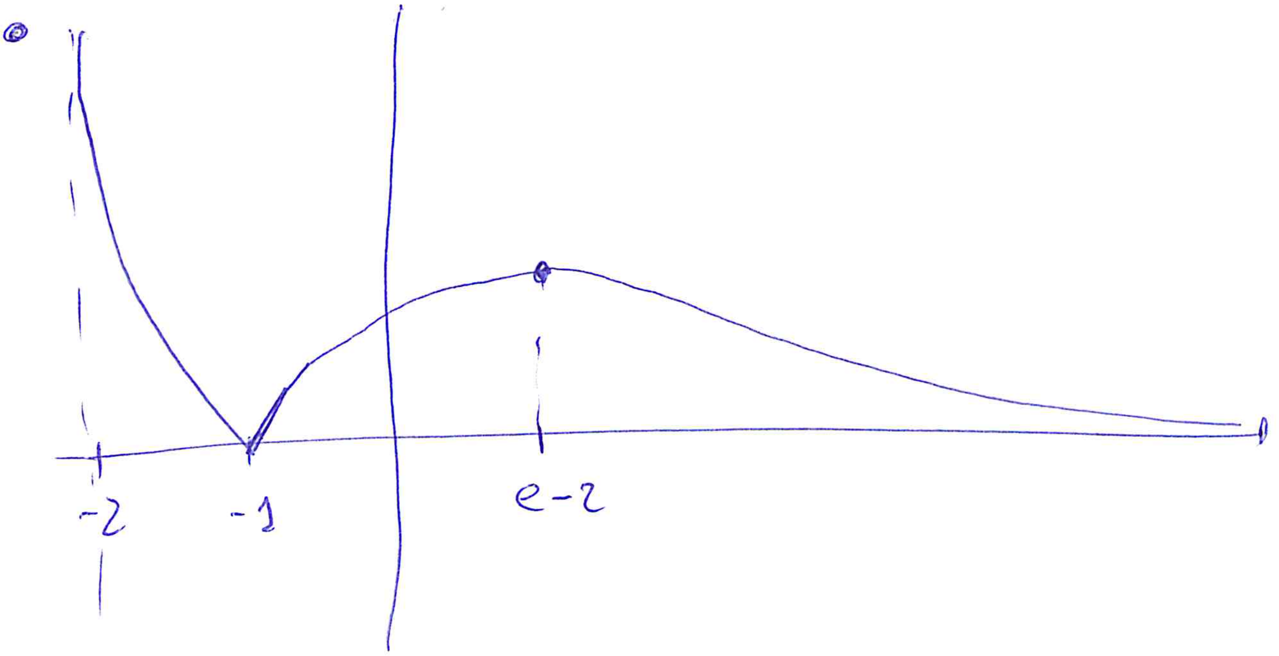
Studiamo il limite di f' per $x \to -1$:

$$\lim_{x \to -1^+} f'(x) = \underbrace{+1}_{\operatorname{sgn}\left(\frac{\log(x+1)}{x+2}\right)} \cdot \left(\frac{1 - \log 1}{3^2}\right) = \frac{1}{9}$$

$$\lim_{x \to -1^-} f'(x) = -1 \cdot \frac{1}{9} = -\frac{1}{9}$$

$\neq$

$\Rightarrow f$ non è derivabile in -1



$\sup f = +\infty$

$\inf f = \min f = 0$

$x = -1$ = p.to di MIN ASSOLUTO

$x = e-2$ = p.to di MAX LOCALE

(A) $f(x) = e^x \sin\left(x + \frac{\pi}{2}\right)$, $g: \mathbb{R} \to \mathbb{R}$ $g'(1) = 2$ (10)

Sia $h(x) = g(f(x))$, calcolare $h'(0)$

$$\boxed{h'(0) = g'(f(0)) \cdot f'(0)}$$

$f(0) = e^0 \cdot \sin\left(\frac{\pi}{2}\right) = 1$

$\Rightarrow h'(0) = g'(1) \cdot f'(0) = 2 f'(0)$

$f'(x) = e^x \cdot \sin\left(x + \frac{\pi}{2}\right) + e^x \cdot \cos\left(x + \frac{\pi}{2}\right)$

$f'(0) = 1 + 0 = 1$

$\Rightarrow \boxed{h'(0) = 2 \cdot 1 = 2}$

(B) $f(x) = \cos(3x^2) \log(x+1)$ $g: \mathbb{R} \to \mathbb{R}$ $g'(0) = 2$

$h'(0) = g'(f(0)) \cdot f'(0)$

$f(0) = 0 \Rightarrow g'(f(0)) = g'(0) = 2$

$f'(x) = -6x \sin(3x^2) \log(x+1) + \frac{\cos(3x^2)}{x+1}$

$f'(0) = 0 + \frac{1}{1} = 1$

$\Rightarrow \boxed{h'(0) = 2 \cdot 1 = 2}$